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On Waring's problem for partially symmetric tensors *(variations on a theme of Mella)*

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Abstract

Here we address a Waring type problem for partially symmetric tensors, extending previous work by Massimiliano Mella in the totally symmetric case of forms. In particular, we provide an explicit answer in lower dimensional cases.

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1 Introduction

Waring's problem for forms, concerning the decomposition of a generic homogeneous polynomial as a sum of powers of linear forms, has now a complete and rigorous solution thanks to the spectacular results obtained by by Alexander and Hirschowitz in [1]. However, several related problems are still widely open: for instance, one would like to determine all cases in which the above decomposition is unique. As far as we know, the best achievement in this direction is the following Theorem, recently proved by Massimiliano Mella (see [10], Theorem 1):

Theorem 1. (Mella) *Fix integers $d > r > 1$ and $k \geq 1$ such that $(k+1)(r+1) = \binom{r+d}{r}$. Then the generic homogeneous polynomial of degree d in $r+1$ variables can be expressed as a sum of $k+1$ d-th powers of linear forms in a unique way if and only if $d = 5$ and $r = 2$.*

It seems rather interesting also to consider variants of the original Waring's problem, the best known one being the so-called Extended Waring's

Problem (see [5], Problem 7.6, and [9]), which asks for a simultaneous decomposition of several forms. Here instead we propose another generalization, which regards forms as symmetric tensors and is concerned with the decomposition of a wider class of tensors. More precisely, for fixed integers $n \geq 1$, $r \geq 1$, and $\underline{d} = (d_1, \ldots, d_n)$, let V_{r,d_i} be the Veronese embedding of $\mathbb{P}^r$ of degree d_i and let $\Sigma_{r,\underline{d}}$ denote the Segre embedding of $V_{r,d_1} \times \ldots \times V_{r,d_n}$ into $\mathbb{P}^N$, $N = \prod_{i=1}^n \binom{r+d_i}{r} - 1$. According to [3], § 4, we can view $\mathbb{P}^N$ inside $\mathbb{P}^M$, $M = (r+1)^{d_1+\ldots+d_n} - 1$, as the space of tensors which are invariant with respect to the natural actions of the symmetric groups $S_{d_1}, \ldots, S_{d_n}$ on the coordinates of $\mathbb{P}^M$. Therefore we identify the points of $\mathbb{P}^N$ with the $(d_1, \ldots, d_n)$ partially symmetric tensors in $\mathbb{P}^M$; in particular, the points on the Segre-Veronese variety $\Sigma_{r,\underline{d}}$ correspond to the partially symmetric tensors in $\mathbb{P}^M$ which are decomposable. Now we can state a general result about decomposition of partially symmetric tensors (we refer to [4] for the definitions of defectivity and weak defectivity):

Theorem 2. *Fix integers $n \geq 1$ and $r \geq 1$ with $nr \geq 2$, $d_n \geq d_{n-1} \geq \ldots \geq d_1 \geq r+1$ and $k \geq 1$ such that*

$$\prod_{i=1}^{n} \binom{r+d_i}{r} = (nr+1)(k+1).$$

If $\Sigma_{r,\underline{d}}$ is neither k-defective nor $(k-1)$-weakly defective, then a general $(d_1, \ldots, d_n)$ partially symmetric tensor in $\mathbb{P}^{(r+1)^{d_1+\ldots+d_n}-1}$ can be expressed in $\nu \geq 2$ ways as a sum of $k+1$ decomposable tensors.

Notice that the special case $n = 1$ of Theorem 2 is the main ingredient of Mella's Theorem 1 (some additional work is needed in order to check the assumption about weak defectivity, see [10], § 3 and § 4). Next we point out a couple of new results, obtained from Theorem 2 together with previous contributions (see [4] and [3]). The easiest one is the following:

Corollary 1. *Fix integers $d_2 \geq d_1 \geq 4$ and $k \geq 1$ such that $(d_1+1)(d_2+1) = 3(k+1)$. Then a general (d_1, d_2) partially symmetric tensor in $\mathbb{P}^{2^{d_1+d_2}-1}$ can be expressed in $\nu \geq 2$ ways as a sum of $k+1$ decomposable tensors.*

A careful application of the so-called Horace method (see Proposition 1 and Proposition 2) yields also the following:

Corollary 2. *Fix integers $d_3 \geq d_2 \geq d_1 \geq 3$ and $k \geq 1$ such that $(d_1+1)(d_2+1) = 4(k+1)$. Assume that*

$$k+1 \leq (d_3-2)\left[\frac{(d_1+1)(d_2+1)}{3}\right] \qquad (1)$$

where $[.]$ *denotes the integral part. Then a general* (d_1, d_2, d_3) *partially symmetric tensor in* $\mathbb{P}^{2^{d_1+d_2+d_3}-1}$ *can be expressed in* $\nu \geq 2$ *ways as a sum of* $k+1$ *decomposable tensors.*

We work over an algebraically closed field of characteristic zero.

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2 The proofs

The proof of Theorem 2 involves two main ingredients.

The first result is well-known to the specialists (indeed, Bronowski was aware of it already in 1932: see [2] for the original statement and [6], Corollary 4.2, [7], Theorem 2.7, [10], Theorem 2.1, for rigorous modern proofs):

Lemma 1. *Let* $X \subset \mathbb{P}^{r(k+1)+k}$ *be a smooth and irreducible projective variety of dimension* r *such that through the general point of* $\mathbb{P}^{r(k+1)+k}$ *there is exactly one* $\mathbb{P}^k$ *which is* $(k+1)$*-secant to* X*. Then the projection of* X *to* $\mathbb{P}^r$ *from a general tangent space to the* $(k-1)$*-secant variety* $S^{k-1}(X)$ *is birational.*

The second result is a version of the so-called Noether-Fano inequality for Mori fiber spaces (see [8], Definition (3.1) and Theorem (4.2)):

Lemma 2. *Let* $\pi : X \to S$ *and* $\rho : Y \to T$ *be two Mori fiber spaces and let* $\varphi : X \dashrightarrow Y$ *be a birational not biregular map. Choose a very ample linear system* $\mathcal{H}_Y$ *on* Y*, define* $\mathcal{H}_X = \varphi^*(\mathcal{H}_Y)$ *and let* $\mu \in \mathbb{Q}$ *such that* $\mathcal{H}_X \equiv -\mu K_X + \varphi^*(A)$ *for some divisor* A *on* S*. Then either* $(X, 1/\mu\mathcal{H}_X)$ *has not canonical singularities or* $K_X + 1/\mu\mathcal{H}_X$ *is not nef.*

Proof of Theorem 2. Since $\Sigma_{r,\underline{d}}$ lies in a projective space $\mathbb{P}^N$ of dimension equal to the expected dimension of the k-secant variety $S^k(\Sigma_{r,\underline{d}})$, the non k-defectivity of $\Sigma_{r,\underline{d}}$ implies that through the general point of $\mathbb{P}^N$ there is at least one $\mathbb{P}^k$ which is $(k+1)$-secant to $\Sigma_{r,\underline{d}}$. Hence the existence of at least one expression as in the statement follows from the translation described in the Introduction and we have only to check that such an expression is never unique. Arguing by contradiction, assume that through the general point of $\mathbb{P}^N$ there is exactly one $\mathbb{P}^k$ which is $(k+1)$-secant to $\Sigma_{r,\underline{d}}$. From Lemma 1 we obtain a birational map

$$\mathbb{P}^r \times \ldots \times \mathbb{P}^r \dashrightarrow \mathbb{P}^{nr}$$

induced by the linear system $\mathcal{H} = \mathcal{O}_{\mathbb{P}^r}(d_1) \otimes \ldots \otimes \mathcal{O}_{\mathbb{P}^r}(d_n)(-2p_1 \ldots - 2p_k)$, where the p_i's are general points in $\mathbb{P}^r \times \ldots \times \mathbb{P}^r$. Since $\Sigma_{r,\underline{d}}$ is not $(k-1)$-weakly defective, by [4], Theorem 1.4, a general divisor $H \in \mathcal{H}$ has only

ordinary double points at $p_1, \ldots, p_k$ and is elsewhere smooth. In particular, H is irreducible and after the blow-up of $p_1, \ldots, p_k$ it becomes smooth, hence we see that H has only canonical singularities. Since both $\mathbb{P}^r \times \ldots \times \mathbb{P}^r$ and $\mathbb{P}^{nr}$ regarded as projective bundles are Mori fiber spaces, we can apply Lemma 2 with $\mu := \frac{d_1}{r+1}$. By our numerical assumptions, we have $-(r+1) + \frac{1}{\mu} d_i \geq 0$ for every i, hence $K + 1/\mu\mathcal{H}$ is nef and this contradiction ends the proof.

□

Proof of Corollary 1. According to Theorem 2, we have only to show that the Segre embedding of $V_{1,d_1} \times V_{1,d_2}$ is neither k-defective nor $(k-1)$-defective. These two properties can be easily checked by looking at the Classification Theorem 1.3 of [4], so the proof is over.

□

Proposition 1. *Fix integers $n \geq 1$, $r_1, \ldots, r_n \geq 1$, $d_1, \ldots, d_n \geq 1$, $d_{n+1} \geq 2$, $0 \leq h \leq \left[\frac{\prod_{i=1}^n \binom{r_i+d_i}{r_i}}{(r_1+\ldots+r_n+1)}\right]$, $0 \leq l \leq \left[\frac{\prod_{i=1}^n \binom{r_i+d_i}{r_i}(d_{n+1}+1)}{(r_1+\ldots+r_n+2)}\right] - \left[\frac{\prod_{i=1}^n \binom{r_i+d_i}{r_i}}{(r_1+\ldots+r_n+1)}\right]$. Choose l general points $p_1, \ldots, p_l$ in $\mathbb{P}^{r_1} \times \ldots \times \mathbb{P}^{r_n} \times \mathbb{P}^1$, let D be a divisor of type $(0, \ldots, 0, 1)$ on $\mathbb{P}^{r_1} \times \ldots \times \mathbb{P}^{r_n} \times \mathbb{P}^1$, and pick h general points $q_1, \ldots, q_h$ on D. Assume that*

$$\dim |\mathcal{O}_{\mathbb{P}^{r_1} \times \ldots \times \mathbb{P}^{r_n} \times \mathbb{P}^1}(d_1, \ldots, d_n, d_{n+1} - 1) \left(-\sum_{i=1}^{l} 2p_i\right)| =$$
$$= \prod_{i=1}^{n} \binom{r_i + d_i}{r_i} (d_{n+1}) - (r_1 + \ldots + r_n + 2)l$$

and

$$\dim |\mathcal{O}_{\mathbb{P}^{r_1} \times \ldots \times \mathbb{P}^{r_n}}(d_1, \ldots, d_n) \left(-\sum_{j=1}^{h} 2q_j\right)| =$$
$$= \prod_{i=1}^{n} \binom{r_i + d_i}{r_i} - (r_1 + \ldots + r_n + 1)h,$$

as expected, and that

$$\dim |\mathcal{O}_{\mathbb{P}^{r_1} \times \ldots \times \mathbb{P}^{r_n} \times \mathbb{P}^1}(d_1, \ldots, d_n, d_{n+1} - 2) \left(-\sum_{i=1}^{l} 2p_i\right)| \leq \prod_{i=1}^{n} \binom{r_i + d_i}{r_i} (d_{n+1})$$
$$-(r_1 + \ldots + r_n + 2)l - h.$$

Then we have

$$\dim |\mathcal{O}_{\mathbb{P}^{r_1} \times \ldots \times \mathbb{P}^{r_n} \times \mathbb{P}^1}(d_1, \ldots, d_n, d_{n+1}) \left(-\sum_{i=1}^{l} 2p_i - \sum_{j=1}^{h} 2q_j\right)| =$$

$$= \prod_{i=1}^{n} \binom{r_i + d_i}{r_i} (d_{n+1} + 1) - (r_1 + \ldots + r_n + 2)(l + h),$$

as expected.

Proof. Let $X := \mathbb{P}^{r_1} \times \ldots \times \mathbb{P}^{r_n} \times \mathbb{P}^1$, $L := \mathcal{O}_X(d_1, \ldots, d_n, d_{n+1})$ and $Z := \cup_{i=1}^{l} 2p_i \cup \cup_{j=1}^{h} 2q_j$; we have to show that $H^1(L \otimes \mathcal{I}_{Z,X}) = 0$. Recall the natural exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z),X} \otimes L(-D) \to \mathcal{I}_{Z,Y} \otimes L \to \mathcal{I}_{Z\cap D,D} \otimes (L_{|D}) \to 0,$$

where $\mathrm{Res}_D(Z) = \cup_{i=1}^{l} 2p_i \cup \cup_{j=1}^{h} q_j$ denotes the residual scheme of Z with respect to D; we are going to apply the usual Horace method by checking that

$$H^1\left(\mathcal{I}_{\mathrm{Res}_D(Z),X} \otimes L(-D)\right) = 0 \tag{2}$$

$$H^1\left(\mathcal{I}_{Z\cap D,D} \otimes (L_{|D})\right) = 0. \tag{3}$$

Since

$$\mathcal{I}_{\mathrm{Res}_D(Z),X} \otimes L(-D) = \mathcal{O}_{\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}\times\mathbb{P}^1}(d_1, \ldots, d_n, d_{n+1}-1)\left(-\sum_{i=1}^{l} 2p_i - \sum_{j=1}^{h} q_j\right),$$

in order to get (2) it is enough to see that the h points q_j on D impose independent conditions on the non-special linear system $|\mathcal{O}_{\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}\times\mathbb{P}^1}(d_1, \ldots, d_n, d_{n+1} - 1)\left(-\sum_{i=1}^{l} 2p_i\right)|$. Indeed, if this were not the case, every divisor in such a linear system passing through $q_1, \ldots, q_{h-1}$ should contain D, hence

$$\dim |\mathcal{O}_{\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}\times\mathbb{P}^1}(d_1, \ldots, d_n, d_{n+1} - 2)\left(-\sum_{i=1}^{l} 2p_i\right)| \geq$$

$$\geq \dim |\mathcal{O}_{\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}}(d_1, \ldots, d_n - 1)\left(-\sum_{i=1}^{l} 2p_i - \sum_{j=1}^{h-1} q_j\right)| \geq$$

$$\geq \prod_{i=1}^{n} \binom{r_i + d_i}{r_i} (d_{n+1}) - (r_1 + \ldots + r_n + 2)l - (h - 1),$$

in contradiction with our assumptions. Finally, notice that

$$\mathcal{I}_{Z\cap D,D} \otimes (L_{|D}) = \mathcal{O}_{\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}}(d_1, \ldots, d_n)\left(-\sum_{j=1}^{h} 2q_j\right),$$

so (3) follows from our assumptions too.

$\square$

Proposition 2. *Fix integers* $n \geq 1$, $r_1, \ldots, r_n \geq 1$, $d_1, \ldots, d_n \geq 2$, $1 \leq s \leq \left[\frac{\prod_{i=1}^{n}\binom{r_i+d_i}{r_i}(d_{n+1}+1)}{(r_1+\ldots+r_n+2)}\right]$, $h_0 := \left[\frac{\prod_{i=1}^{n}\binom{r_i+d_i}{r_i}}{(r_1+\ldots+r_n+1)}\right]$, t_0 *such that* $1 \leq s - t_0 h_0 \leq h_0$, *and* $d_{n+1} \geq t_0 + 3$. *Assume that*

$$\dim |\mathcal{O}_{\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}\times\mathbb{P}^1}(d_1, \ldots, d_n, d_{n+1} - t + 1)\left(-\sum_{i=1}^{s-th_0} 2p_i - \sum_{j=1}^{h_0} 2q_j\right)| =$$

$$= \prod_{i=1}^{n}\binom{r_i+d_i}{r_i}(d_{n+1} - t + 2) - (r_1 + \ldots + r_n + 2)(s - (t-1)h_0)$$

for every $1 \leq t \leq t_0$, *and*

$$\dim |\mathcal{O}_{\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}\times\mathbb{P}^1}(d_1, \ldots, d_n, d_{n+1} - t_0)\left(-2p_1 - \sum_{j=1}^{s-t_0h_0-1} 2q_j\right)| =$$

$$= \prod_{i=1}^{n}\binom{r_i+d_i}{r_i}(d_{n+1} - t_0 + 1) - (r_1 + \ldots + r_n + 2)(s - t_0h_0),$$

where the p_i*'s are general points in* $\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}\times\mathbb{P}^1$ *and the* q_i*'s are general points on a divisor* D *of type* $(0, \ldots, 0, 1)$. *Then a general divisor in the linear system*

$$|\mathcal{O}_{\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}\times\mathbb{P}^1}(d_1, \ldots, d_n, d_{n+1})\left(-\sum_{i=1}^{s} 2p_i\right)| \tag{4}$$

has only ordinary double points at $p_1, \ldots, p_s$ *and is elsewhere smooth. In particular, the Segre embedding of* $V_{r_1,d_1}\times\ldots\times V_{r_n,d_n}\times V_{1,d_{n+1}}$ *is not* $(s-1)$*-weakly defective.*

Proof. Since the linear system

$$|\mathcal{O}_{\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}\times\mathbb{P}^1}(d_1, \ldots, d_n, d_{n+1})\left(-\sum_{i=1}^{s-h_0} 2p_i - \sum_{j=1}^{h_0} 2q_j\right)|$$

has the expected dimension, a general divisor in (4) specializes to $E + D$, where E is a general divisor in

$$|\mathcal{O}_{\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}\times\mathbb{P}^1}(d_1, \ldots, d_n, d_{n+1} - 1)\left(-\sum_{i=1}^{s-h_0} 2p_i\right)|.$$

Hence by [4], Theorem 1.4, we are reduced to prove that E has only ordinary double points. Moreover, by repeating t_0 times exactly the same argument, we are reduced to prove that the general divisor F in

$$|\mathcal{O}_{\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}\times\mathbb{P}^1}(d_1,\ldots,d_n,d_{n+1}-t_0)\left(-\sum_{i=1}^{s-t_0h_0}2p_i\right)|$$

has only ordinary double points. Once again, let F degenerate to $G+D$, where G is a general divisor in

$$|\mathcal{O}_{\mathbb{P}^{r_1}\times\ldots\times\mathbb{P}^{r_n}\times\mathbb{P}^1}(d_1,\ldots,d_n,d_{n+1}-t_0-1)\,(-2p_1)\,|.$$

We claim that G has an isolated ordinary double point in p_1. Indeed, by [4], Theorem 1.4, it is enough to check that the Segre embedding of $V_{r_1,d_1}\times\ldots\times V_{r_n,d_n}\times V_{1,d_{n+1}-t_0-1}$ is not 0-weakly defective, but this is clear, since by [4], Remark 3.1 (ii), 0-weakly defective varieties contain many lines, here instead we have $d_i\geq 2$ for $i=1,\ldots,n-1$, and $d_{n+1}-t_0-1\geq 2$. Hence the claim is established and we conclude by [4], Theorem 1.4.

□

Proof of Corollary 2. By [3], Theorem 2.1 and Theorem 2.5, if $d_i\geq 3$ for $i=1,2$, then both the linear systems $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d_1,d_2)(-\sum 2p_i)$ and $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(d_1,d_2,d_3)(-\sum 2p_i)$ have always the expected dimension. In particular, $\Sigma_{1,\underline{d}}$ is never k-defective and for every integer t such that $t\leq d_3-3$ we have

$$\begin{aligned}\dim|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(d_1,d_2,d_3-t-2)\left(-\sum_{i=1}^{l}2p_i\right)|\\ =(d_1+1)(d_2+1)(d_3-t+1)-4l\\ <(d_1+1)(d_2+1)(d_3-t+1)-4l-h\end{aligned}$$

for $h<(d_1+1)(d_2+1)$, so all assumptions of Proposition 1 are satisfied. As a consequence, we can apply Proposition 2 (just notice that $t_0\leq d_3-3$ by (1)), and deduce that $\Sigma_{1,\underline{d}}$ is not $(k-1)$-defective. Now the claim follows from Theorem 2.

□

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